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Strong Approximation by Fourier Series

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DEDICATED TO THE MEMORY OF GÉZA FREUD

G. Freud had wide research interests which included Fourier series. In this survey paper we intend to show the great influence of his only paper [2] on strong approximation of Fourier series. This article has been the origin of a new subject called nowadays "converse-type results for strong approximation of Fourier series."

Before stating his initial result we outline briefly the background of the subject.

After the classical result of Fejér in 1904 on the convergence of the arithmetical mean of the partial sums of a Fourier series of 2π-periodic functions, Hardy and Littlewood [3] began to investigate the problem of so-called strong summability. It turned out that under certain conditions not only the means

$$\frac{1}{n+1}\sum_{k=0}^{n}(s_k(f;x)-f(x))$$

but also the "stronger" means

$$h_n(f;p;x):=\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(f;x)-f(x)|^p\right\}^{1/p} \tag{1}$$

tend to zero, where $s_k(x)=s_k(f;x)$ denotes the kth partial sum of the Fourier series of $f(x)$. The interest of their result is that it shows, among others, that when the Fourier series of a continuous function is not convergent, its summability is not merely a consequence of the cancellation of the various deviations summed in Fejér's mean, but rather of the comparative smallness of the deviations. Later a number of distinguished mathematicians joined the research of Hardy and Littlewood (e.g., Fejér, Fekete, Carleman, Sutton, Marcinkiewicz, Zygmund). Nevertheless, the problem of the so-called strong approximation (i.e., how fast the means in

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(1) tend to zero) was not investigated for 50 years, in contrast to the ordinary approximation which was very popular.

It was Alexits who began to deal with strong approximation of Fourier series. In a joint paper Alexits and Králik [1] proved the following generalization of the classical Bernstein result: If $f \in \mathrm{Lip}\,\alpha$, $0<\alpha<1$, then

$$h_n(f; 1; x) = O(n^{-\alpha}). \tag{2}$$

Alexits, partly with co-authors, proved several interesting theorems and raised many problems in connection with strong approximation of Fourier series. Under his influence these investigations grew to full proportions; and nowadays the number of papers in this field is near hundred.

The first results (but also some later ones) were so-called direct statements, i.e. they dealt with the problem of how fast a given function class can be approximated by certain strong means; e.g., we have seen such a theorem under (2). In this topic the author of the present paper proved many results for the W^rH^α class (e.g., [6–9]). These results have been generalized for the W^rH^ω class by Totik (e.g., [19–21]). Results of this type were stated mainly for the $C(0, 2\pi)$ space, but recently there are investigations for $L^p(0, 2\pi)$, or even more general spaces, as well as for functions of several variables.

The first converse type result for strong approximation is due to Freud [2]. The converse results are those which deduce properties of the function approximated from the order of strong approximation.

To recall the result of Freud we have to go into some details. In [6] we proved, among others, the following results:

If f has a continuous rth derivative and $f^{(r)} \in \mathrm{Lip}\,\alpha$, then for any positive p and $\beta > (r+\alpha)\,p$

$$\begin{aligned} h_n(f, p, \beta; x) &:= \left\{ \frac{1}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1}\, |s_k(x) - f(x)|^p \right\}^{1/p} \\ &= O(n^{-r-\alpha}); \end{aligned} \tag{3}$$

and if $\beta = (r+\alpha)\,p$ then

$$h_n(f, p, \beta; x) = O\left(\frac{(\log n)^{1/p}}{n^{r+\alpha}} \right). \tag{4}$$

We also showed that estimation (4) cannot be improved in the case $\beta = (r+\alpha)\,p$; since there exists a function f_0 such that $f_0^{(r)} \in \mathrm{Lip}\,\alpha$ and

$$h_n(f_0, p, \beta; 0) \geqslant c\, \frac{(\log n)^{1/p}}{n^{r+\alpha}} \tag{5}$$

with a positive c.

In the special case $\beta=1$, $r=0$ and $\alpha=1/p$, estimations (4) and (5) show that for the whole class Lip $1/p$ we cannot give an estimation better than

$$\left\{\frac{1}{n}\sum_{k=1}^{n}|s_k(x)-f(x)|^p\right\}^{1/p}=O\left(\frac{(\log n)^{1/p}}{n^{1/p}}\right).$$

In connection with this result, in 1969 Freud [2] investigated the following question: If we know that a function f has the property

$$\left\{\frac{1}{n}\sum_{k=1}^{n}|s_k(x)-f(x)|^p\right\}^{1/p}\leqslant\frac{K}{n^{1/p}},\qquad p>1, \tag{6}$$

for all x, what can we say about f?

He proved that (6) implies $f\in$ Lip $1/p$; moreover

$$f(x+h)-f(x)=o_x(h^{1/p}) \tag{7}$$

holds almost everywhere. The proof of these statements does not use any special trick, it is a straightforward calculation.

This result of Freud has its merit in that it opened a new research area. In fact this subject initiated by Freud has become a very interesting and important field of the theory of strong approximation by Fourier series. As we have already mentioned, this theme, to deduce structural properties of the functions approximated by certain strong means at a given order, is called "converse type results."

In the same paper Freud raised one more problem: Does (6) imply (7) for all x?

The negative answer was given by me in [9]; i.e., there exists a function f^* such that estimation (7) is not fulfilled at $x=0$, but (6) holds. The counterexample was

$$f^*(x)=\sum_{n=1}^{\infty}\frac{\sin nx}{n^{1+1/p}}\qquad (p>1).$$

These results have been generalized into several directions. First, we generalized the above results to such strong means which are determined by a general triangular matrix, and we also investigated the case $p=1$ (cf. [10, 11]).

We mention that assumption (6) is obviously equivalent to

$$\sum_{k=1}^{\infty}|s_k(x)-f(x)|^p\leqslant K; \tag{8}$$

and this condition seems to be more lucidly arranged than (6), therefore the assumptions of the converse type results are formulated in the form of (8) instead of (6).

In a joint paper of Leindler and Nikisin [15] it was proved that condition (8) with $p=1$ implies that

$$|f(x+h)-f(x)| \leqslant K_1 h \log \frac{1}{h} \tag{9}$$

holds for all x; and

$$|f(x+h)-f(x)| = O_x(h) \tag{10}$$

for almost all x. Furthermore, there exists a function f_0 satisfying (8) with $p=1$ but

$$f_0\left(\frac{\pi}{2^n}\right) - f_0(0) > \frac{1}{8}\frac{\pi}{2^n}\log\frac{2^n}{\pi} \tag{11}$$

for all $n \geqslant 6$.

Estimation (11) shows that (9) is best possible; and (10) does not hold for all x.

Inequality (11) also shows that condition (8) with $p=1$ does not imply $f \in \mathrm{Lip}\ 1$. In connection with this fact we (see [12, 13]) raised the following problem: Does condition (8) with $0<p<1$ imply $f \in \mathrm{Lip}\ 1$?

The affirmative answer was given only four years later but then simultaneously by two authors, namely Oskolkov [16] and Szabados [17] independently proved the following very fine result which shows that even a weaker condition than (8) with $0<p<1$ yields $f \in \mathrm{Lip}\ 1$.

Their theorem states: Let $\Omega(\delta)$ be an arbitrary modulus of continuity. If

$$\left\| \sum_{n=1}^{\infty} \Omega(|s_n - f|) \right\|_{\sup} < \infty \tag{12}$$

and

$$\int_0^1 \frac{dx}{\Omega(x)} < \infty \tag{13}$$

then $f \in \mathrm{Lip}\ 1$.

It is clear that if $\Omega(\delta)=\delta^p$ with $0<p<1$ then (13) is fulfilled, and (12) reduces to (8), thus this theorem answers our problem.

Under certain restrictions on $\Omega(\delta)$ Oskolkov and Szabados also proved the necessity of condition (13). Regarding the additional conditions on Ω, they differ, e.g., Oskolkov claims the condition:

$$\overline{\lim_{\delta \to +0}} \frac{\Omega(\delta/2)}{\Omega(\delta)} < 1.$$

At that time it seemed to be a difficult task to prove the necessity of (13) without any additional condition. Notwithstanding that in 1978 Totik [18] proved the necessity of (13) without any additional assumption with grace, we can say that our guess was right, namely Totik's proof is hard and ingenious.

In the same year, in a joint paper of Krotov and Leindler [5], a theorem of new type appeared. One of its special cases also gives a sufficient condition in order that f should belong to Lip 1; and it also shows that (8) with $0<p<1$ implies $f \in \mathrm{Lip}\, 1$. This special case (cf. [5]) reads as follows:

Let $\{\lambda_n\}$ be a positive monotone (nondecreasing or nonincreasing) sequence and $0<p<\infty$. Then

$$\left\| \sum_{n=1}^{\infty} \lambda_n |s_n - f|^p \right\| < \infty$$

and

$$\sum_{n=1}^{\infty} (n\lambda_n)^{-1/p} < \infty$$

imply that f belongs to the class Lip 1.

Most of the results mentioned hitherto were generalized to the derivative functions. Now we recall only one of them.

We have seen that condition (8) with $0<p<1$ is too strong to be an exact sufficient condition for $f \in \mathrm{Lip}\, 1$. It implies a stronger property of f than only belonging to the class Lip 1. This was already observed by Szabados [17], but the final result was proved in [14]. This reads as follows:

Let $0<p\leqslant 1$ and $1/p = r+\alpha$, where r is an integer and $0<\alpha\leqslant 1$. If (8) is fulfilled then

$$\omega(f^{(r)}; \delta) = \begin{cases} o\left(\delta \log \dfrac{1}{\delta}\right) & \text{if } \alpha = 1 \\ o(\delta^{\alpha}) & \text{if } 0<\alpha<1, \end{cases}$$

holds, and if $\alpha = 1$, then $f^{(r)}$ belongs to the Zygmund class (here $\omega(g, \delta)$ denotes the modulus of continuity of g).

Furthermore these estimations are best possible.

It would be very easy to recall a number of further interesting results having a converse type character and emphasizing thereby the effect of the cited paper of Freud. However, we mention only one more result due to Totik, which investigates the structural properties of functions arising from the "generalized" strong approximation.

Condition (8) can be easily generalized by replacing the function x^p by a more general function $\Omega(x)$ (see, e.g., (12)) and then one may ask: What differentiability or continuity properties of f can one infer from the fact that f satisfies an assumption of such generalized type? As we have seen the first result of this type is due to Oskolkov [16] and Szabados [17]. The most general results of this type were proved by Totik [19–21]. Some further theorems of similar character are due to Krotov [4].

In [20] Totik proved the following theorem:

Let Ω be a convex or concave function with the properties:

$$\Omega(x)>0 \quad (x>0), \qquad \lim_{x \to +0} \Omega(x)=\Omega(0)=0,$$

furthermore let $\{\lambda_n\}$, $\{\mu_n\}$ be positive nondecreasing sequences. If

$$\left\| \sum_{n=0}^{\infty} \lambda_n \Omega(\mu_n |s_n - f|) \right\| < \infty$$

then

$$\omega\left(f, \frac{1}{n}\right) = O\left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{\mu_k} \bar{\Omega}\left(\frac{1}{k\lambda_k}\right)\right), \tag{14}$$

where $\bar{\Omega}$ denotes the inverse of Ω.

Estimation (14) is, in general, the best possible.

Acknowledgment

I hope this short survey demonstrates the deep mathematical insight of Freud in choosing problems; and my high esteem of the work of my admirable Hungarian colleague.

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